

# Well-posedness for the Navier-Stokes equations with data in homogeneous Sobolev-Lorentz spaces

D. Q. Khai, N. M. Tri

Institute of Mathematics, VAST  
18 Hoang Quoc Viet, 10307 Cau Giay, Hanoi, Vietnam

**Abstract:** In this paper, we study local well-posedness for the Navier-Stokes equations (NSE) with the arbitrary initial value in homogeneous Sobolev-Lorentz spaces  $\dot{H}_{L^{q,r}}^s(\mathbb{R}^d) := (-\Delta)^{-s/2} L^{q,r}$  for  $d \geq 2, q > 1, s \geq 0, 1 \leq r \leq \infty$ , and  $\frac{d}{q} - 1 \leq s < \frac{d}{q}$ , this result improves the known results for  $q > d, r = q, s = 0$  (see [4, 7]) and for  $q = r = 2, \frac{d}{2} - 1 < s < \frac{d}{2}$  (see [4, 9]). In the case of critical indexes ( $s = \frac{d}{q} - 1$ ), we prove global well-posedness for NSE provided the norm of the initial value is small enough. The result that is a generalization of the result in [5] for  $q = r = d, s = 0$ .

## §1. Introduction

We consider the Navier-Stokes equations in  $\mathbb{R}^d$ :

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \otimes u) - \nabla p, \\ \nabla \cdot u = 0, \\ u(0, x) = u_0, \end{cases}$$

which is a condensed writing for

$$\begin{cases} 1 \leq k \leq d, & \partial_t u_k = \Delta u_k - \sum_{l=1}^d \partial_l (u_l u_k) - \partial_k p, \\ \sum_{l=1}^d \partial_l u_l = 0, \\ 1 \leq k \leq d, & u_k(0, x) = u_{0k}. \end{cases}$$

The unknown quantities are the velocity  $u(t, x) = (u_1(t, x), \dots, u_d(t, x))$  of the fluid element at time  $t$  and position  $x$  and the pressure  $p(t, x)$ .

In the 1960s, mild solutions were first constructed by Kato and Fujita ([18], [19]) that are continuous in time and take values in the Sobolev spaces

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<sup>3</sup>*e-mail address*: khaitoantin@gmail.com triminh@math.ac.vn

$H^s(\mathbb{R}^d)$ , ( $s \geq \frac{d}{2} - 1$ ), say  $u \in C([0, T]; H^s(\mathbb{R}^d))$ . In 1992, a modern treatment for mild solutions in  $H^s(\mathbb{R}^d)$ , ( $s \geq \frac{d}{2} - 1$ ) was given by Chemin [9]. In 1995, using the simplified version of the bilinear operator, Cannone proved the existence of mild solutions in  $\dot{H}^s(\mathbb{R}^d)$ , ( $s \geq \frac{d}{2} - 1$ ), see [4]. Results on the existence of mild solutions with value in  $L^q(\mathbb{R}^d)$ , ( $q > d$ ) were established in the papers of Fabes, Jones and Rivière [11] and of Giga [14]. Concerning the initial data in the space  $L^\infty$ , the existence of a mild solution was obtained by Cannone and Meyer in ([4], [7]). In 1994, Kato and Ponce [23] showed that the NSE are well-posed when the initial data belong to the homogeneous Sobolev spaces  $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ , ( $d \leq q < \infty$ ). Recently, the authors of this article have considered NSE in the mixed-norm Sobolev-Lorentz spaces, see [17]. In this paper, for  $d \geq 2$ ,  $q > 1$ ,  $s \geq 0$ ,  $1 \leq r \leq \infty$ , and  $\frac{d}{q} - 1 \leq s < \frac{d}{q}$ , we investigate mild solutions to NSE in the spaces  $L^\infty([0, T]; \dot{H}_{L^{q,r}}^s(\mathbb{R}^d))$  when the initial data belong to the Sobolev-Lorentz spaces  $\dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$ , which are more general than the spaces  $\dot{H}_q^s(\mathbb{R}^d)$ , ( $\dot{H}_q^s(\mathbb{R}^d) = \dot{H}_{L^{q,q}}^s(\mathbb{R}^d)$ ). We obtain the existence of mild solutions with arbitrary initial value when  $T$  is small enough, and existence of mild solutions for any  $T > 0$  when the norm of the initial value in the Besov spaces  $\dot{B}_{\tilde{q}}^{s-d(\frac{1}{q}-\frac{1}{\tilde{q}}),\infty}(\mathbb{R}^d)$ , ( $\frac{1}{2}(\frac{1}{q} + \frac{s}{d}) < \frac{1}{\tilde{q}} < \min\{\frac{1}{2} + \frac{s}{2d}, \frac{1}{q}\}$ ) is small enough.

In the particular case ( $q > d, r = q, s = 0$ ), we get the result which is more general than that of Cannone and Meyer ([4], [7]). Here we obtained a statement that is stronger than that of Cannone and Meyer but under a much weaker condition on the initial data.

In the particular case ( $q = r = 2, \frac{d}{2} - 1 < s < \frac{d}{2}$ ), we get the result which is more general than those of Chemin in [9] and Cannone in [4]. Here we obtained a statement that is stronger than those of Chemin in [9] and Cannone in [4] but under a much weaker condition on the initial data.

In the case of critical indexes ( $1 < q \leq d, r \geq 1, s = \frac{d}{q} - 1$ ), we get a result that is a generalization of a result of Cannone [5]. In particular, when  $q = r = d, s = 0$ , we get back the Cannone theorem (Theorem 1.1 in [5]).

The paper is organized as follows. In Section 2 we prove some inequalities for pointwise products in the Sobolev spaces and some auxiliary lemmas. In Section 3 we present the main results of the paper. In the sequence, for a space of functions defined on  $\mathbb{R}^d$ , say  $E(\mathbb{R}^d)$ , we will abbreviate it as  $E$ .

## §2. Some auxiliary results

In this section, we recall the following results and notations.

**Definition 1.** (Lorentz spaces). (See [1].)

For  $1 \leq p, r \leq \infty$ , the Lorentz space  $L^{p,r}(\mathbb{R}^d)$  is defined as follows: A measurable function  $f \in L^{p,r}(\mathbb{R}^d)$  if and only if

$$\|f\|_{L^{p,r}(\mathbb{R}^d)} := \left( \int_0^\infty (t^{\frac{1}{p}} f^*(t))^r \frac{dt}{t} \right)^{\frac{1}{r}} < \infty \text{ when } 1 \leq r < \infty,$$

$$\|f\|_{L^{p,\infty}(\mathbb{R}^d)} := \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty \text{ when } r = \infty,$$

where  $f^*(t) = \inf \{ \tau : \mathcal{M}^d(\{x : |f(x)| > \tau\}) \leq t \}$ , with  $\mathcal{M}^d$  being the Lebesgue measure in  $\mathbb{R}^d$ .

Before proceeding to the definition of Sobolev-Lorentz spaces, let us introduce several necessary notations. For real number  $s$ , the operator  $\dot{\Lambda}^s$  is defined through Fourier translation by

$$(\dot{\Lambda}^s f)^\wedge(\xi) = |\xi|^s \hat{f}(\xi).$$

For  $0 < s < d$ , the operator  $\dot{\Lambda}^s$  can be viewed as the inverse of the Riesz potential  $I_s$  up to a positive constant

$$I_s(f)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-s}} dy \text{ for } x \in \mathbb{R}^d.$$

For  $q > 1, r \geq 1$ , and  $0 \leq s < \frac{d}{q}$ , the operator  $I_s$  is continuous from  $L^{q,r}$  to  $L^{\tilde{q},r}$ , where  $\frac{1}{\tilde{q}} = \frac{1}{q} - \frac{s}{d}$ , see ([26], Theorem 2.4 *iii*), p. 20).

**Definition 2.** (Sobolev-Lorentz spaces). (See [12].)

For  $q > 1, r \geq 1$ , and  $0 \leq s < \frac{d}{q}$ , the Sobolev-Lorentz space  $\dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$  is defined as the space  $I_s(L^{q,r}(\mathbb{R}^d))$ , equipped with the norm

$$\|f\|_{\dot{H}_{L^{q,r}}^s} := \|\dot{\Lambda}^s f\|_{L^{q,r}}.$$

**Lemma 1.** Let  $q > 1, 1 \leq r \leq \tilde{r} \leq \infty$ , and  $0 \leq s < \frac{d}{q}$ . Then we have the following imbedding maps

(a)

$$\dot{H}_{L^{q,1}}^s \hookrightarrow \dot{H}_{L^{q,r}}^s \hookrightarrow \dot{H}_{L^{q,\tilde{r}}}^s \hookrightarrow \dot{H}_{L^{q,\infty}}^s.$$

(b)  $\dot{H}_q^s = \dot{H}_{L^{q,q}}^s$  (equality of the norm).

**Proof.** It is easily deduced from the properties of the standard Lorentz spaces.  $\square$

In the following lemmas, we estimate the pointwise product of two functions in  $\dot{H}_q^s(\mathbb{R}^d)$ , ( $d \geq 2$ ) which is a generalization of the Holder inequality. In the case when  $s = 0$  we get back the usual Holder inequality. Pointwise multiplication results for Sobolev spaces are also obtained in literature, see for example [10], [26], [22] and the references therein.

**Lemma 2.** *Assume that*

$$1 < p, q < d, \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{d}.$$

*Then the following inequality holds*

$$\|uv\|_{\dot{H}_r^1} \lesssim \|u\|_{\dot{H}_p^1} \|v\|_{\dot{H}_q^1}, \quad \forall u \in \dot{H}_p^1, v \in \dot{H}_q^1,$$

where  $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{1}{d}$ .

**Proof.** By applying the Leibniz formula for the derivatives of a product of two functions, we have

$$\|uv\|_{\dot{H}_r^1} \simeq \sum_{|\alpha|=1} \|\partial^\alpha(uv)\|_{L^r} \leq \sum_{|\alpha|=1} \|(\partial^\alpha u)v\|_{L^r} + \sum_{|\alpha|=1} \|u(\partial^\alpha v)\|_{L^r}.$$

By applying the Hölder and Sobolev inequalities we obtain

$$\sum_{|\alpha|=1} \|(\partial^\alpha u)v\|_{L^r} \leq \sum_{|\alpha|=1} \|\partial^\alpha u\|_{L^p} \|v\|_{L^{q_1}} \lesssim \|u\|_{\dot{H}_p^1} \|v\|_{\dot{H}_q^1},$$

where

$$\frac{1}{q_1} = \frac{1}{q} - \frac{1}{d}.$$

Similar to the above reasoning, we have

$$\sum_{|\alpha|=1} \|u(\partial^\alpha v)\|_{L^r} \lesssim \|u\|_{\dot{H}_p^1} \|v\|_{\dot{H}_q^1}.$$

This gives the desired result

$$\|uv\|_{\dot{H}_r^1} \lesssim \|u\|_{\dot{H}_p^1} \|v\|_{\dot{H}_q^1}.$$

□

**Lemma 3.** *Assume that*

$$0 \leq s \leq 1, \frac{1}{p} > \frac{s}{d}, \frac{1}{q} > \frac{s}{d}, \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \frac{s}{d}. \quad (1)$$

*Then the following inequality holds*

$$\|uv\|_{\dot{H}_r^s} \lesssim \|u\|_{\dot{H}_p^s} \|v\|_{\dot{H}_q^s}, \quad \forall u \in \dot{H}_p^s, v \in \dot{H}_q^s,$$

where  $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{s}{d}$ .

**Proof.** It is not difficult to show that if  $p, q$ , and  $s$  satisfy (1) then there exists numbers  $p_1, p_2, q_1, q_2 \in (1, +\infty)$  (may be many of them) such that

$$\frac{1}{p} = \frac{1-s}{p_1} + \frac{s}{p_2}, \frac{1}{q} = \frac{1-s}{q_1} + \frac{s}{q_2}, \frac{1}{p_1} + \frac{1}{q_1} < 1,$$

$$p_2 < d, q_2 < d, \text{ and } \frac{1}{p_2} + \frac{1}{q_2} < 1 + \frac{1}{d}.$$

Setting

$$\frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{q_1}, \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{d},$$

we have

$$\frac{1}{r} = \frac{1-s}{r_1} + \frac{s}{r_2}.$$

Therefore, applying Theorem 6.4.5 (page 152) of [1] (see also [25] for  $\dot{H}_p^s$ ), we get

$$\dot{H}_p^s = [L^{p_1}, \dot{H}_{p_2}^1]_s, \dot{H}_q^s = [L^{q_1}, \dot{H}_{q_2}^1]_s, \dot{H}_r^s = [L^{r_1}, \dot{H}_{r_2}^1]_s.$$

Applying the Holder inequality and Lemma 2 in order to obtain

$$\|uv\|_{L^{r_1}} \lesssim \|u\|_{L^{p_1}} \|v\|_{L^{q_1}}, \quad \forall u \in L^{p_1}, v \in L^{q_1},$$

$$\|uv\|_{\dot{H}_{r_2}^1} \lesssim \|u\|_{\dot{H}_{p_2}^1} \|v\|_{\dot{H}_{q_2}^1}, \quad \forall u \in \dot{H}_{p_2}^1, v \in \dot{H}_{q_2}^1.$$

From Theorem 4.4.1 (page 96) of [1] we get

$$\|uv\|_{\dot{H}_r^s} \lesssim \|u\|_{\dot{H}_p^s} \|v\|_{\dot{H}_q^s}.$$

□

**Lemma 4.** Assume that

$$q > 1, p > 1, 0 \leq \frac{s}{d} < \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}, \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \frac{s}{d}. \quad (2)$$

Then we have the inequality

$$\|uv\|_{\dot{H}_r^s} \lesssim \|u\|_{\dot{H}_p^s} \|v\|_{\dot{H}_q^s}, \quad \forall u \in \dot{H}_p^s, v \in \dot{H}_q^s,$$

where  $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{s}{d}$ .

**Proof.** Denote by  $[s]$  the integer part of  $s$  and by  $\{s\}$  the fraction part of the argument  $s$ . Using the formula for the derivatives of a product of two functions, we have

$$\begin{aligned} \|uv\|_{\dot{H}_r^s} &= \|\dot{\Lambda}^s(uv)\|_{L^r} = \|\dot{\Lambda}^{\{s\}}(uv)\|_{\dot{H}_r^{[s]}} \simeq \\ &\sum_{|\alpha|=[s]} \|\partial^\alpha \dot{\Lambda}^{\{s\}}(uv)\|_{L^r} = \sum_{|\alpha|=[s]} \|\dot{\Lambda}^{\{s\}} \partial^\alpha(uv)\|_{L^r} \\ &= \sum_{|\alpha|=[s]} \|\partial^\alpha(uv)\|_{\dot{H}_r^{\{s\}}} \lesssim \sum_{|\gamma|+|\beta|=[s]} \|\partial^\gamma u \partial^\beta v\|_{\dot{H}_r^{\{s\}}}. \end{aligned}$$

Set

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{s - |\gamma| - \{s\}}{d}, \quad \frac{1}{\tilde{q}} = \frac{1}{q} - \frac{s - |\beta| - \{s\}}{d}.$$

Applying Lemma 3 and the Sobolev inequality in order to obtain

$$\|\partial^\gamma u \partial^\beta v\|_{\dot{H}_r^{\{s\}}} \lesssim \|\partial^\gamma u\|_{\dot{H}_p^{\{s\}}} \|\partial^\beta v\|_{\dot{H}_q^{\{s\}}} \lesssim \|u\|_{\dot{H}_p^{|\gamma|+\{s\}}} \|v\|_{\dot{H}_q^{|\beta|+\{s\}}} \lesssim \|u\|_{\dot{H}_p^s} \|v\|_{\dot{H}_q^s}.$$

This gives the desired result

$$\|uv\|_{\dot{H}_r^s} \lesssim \|u\|_{\dot{H}_p^s} \|v\|_{\dot{H}_q^s}.$$

□

**Lemma 5.** Let  $1 \leq p, q \leq \infty$  and  $s \in \mathbb{R}$ .

(a) If  $s < 1$  then the two quantities

$$\left( \int_0^\infty (t^{-\frac{s}{2}} \|e^{t\Delta} t^{\frac{1}{2}} \dot{\Lambda} f\|_q)^p \frac{dt}{t} \right)^{1/p} \text{ and } \|f\|_{\dot{B}_q^{s,p}}$$

(b) If  $s < 0$  then the two quantities

$$\left( \int_0^\infty (t^{-\frac{s}{2}} \|e^{t\Delta} f\|_q)^p \frac{dt}{t} \right)^{1/p} \text{ and } \|f\|_{\dot{B}_q^{s,p}}$$

where  $\dot{B}_q^{s,p}$  is the homogeneous Besov space.

**Proof.** See ([13], Proposition 1, p. 181 and Proposition 3, p. 182), or see ([26], Theorem 5.4, p. 45). □

The following lemma is a generalization of the above lemma.

**Lemma 6.** Let  $1 \leq p, q \leq \infty$ ,  $\alpha \geq 0$ , and  $s < \alpha$ . Then the two quantities

$$\left( \int_0^\infty (t^{-\frac{s}{2}} \|e^{t\Delta} t^{\frac{\alpha}{2}} \dot{\Lambda}^\alpha f\|_{L^q})^p \frac{dt}{t} \right)^{\frac{1}{p}} \text{ and } \|f\|_{\dot{B}_q^{s,p}}$$

**Proof.** Note that  $\dot{\Lambda}^{s_0}$  is an isomorphism from  $\dot{B}_q^{s,p}$  to  $\dot{B}_q^{s-s_0,p}$ , see [3], then we can easily prove the lemma.  $\square$

**Lemma 7.** Assume that  $q > 1, 1 \leq r \leq \infty$ , and  $0 \leq s < \frac{d}{q}$ . The following statement is true: If  $u_0 \in \dot{H}_{L^{q,r}}^s$  then  $e^{t\Delta}u_0 \in L^\infty([0, \infty); \dot{H}_{L^{q,r}}^s)$  and  $\|e^{t\Delta}u_0\|_{L^\infty([0, \infty); \dot{H}_{L^{q,r}}^s)} \leq \|u_0\|_{\dot{H}_{L^{q,r}}^s}$ .

**Proof.** We have

$$\begin{aligned} \|e^{t\Delta}u_0\|_{\dot{H}_{L^{q,r}}^s} &= \|e^{t\Delta}\dot{\Lambda}^s u_0\|_{L^{q,r}} = \frac{1}{(4\pi t)^{d/2}} \left\| \int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}} \dot{\Lambda}^s u_0(\cdot - \xi) d\xi \right\|_{L^{q,r}} \\ &\leq \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}} \|\dot{\Lambda}^s u_0(\cdot - \xi)\|_{L^{q,r}} d\xi \\ &= \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}} \|u_0\|_{\dot{H}_{L^{q,r}}^s} d\xi = \|u_0\|_{\dot{H}_{L^{q,r}}^s}. \end{aligned}$$

$\square$

Let us recall following result on solutions of a quadratic equation in Banach spaces (Theorem 22.4 in [26], p. 227).

**Theorem 1.** Let  $E$  be a Banach space, and  $B : E \times E \rightarrow E$  be a continuous bilinear map such that there exists  $\eta > 0$  so that

$$\|B(x, y)\| \leq \eta \|x\| \|y\|,$$

for all  $x$  and  $y$  in  $E$ . Then for any fixed  $y \in E$  such that  $\|y\| \leq \frac{1}{4\eta}$ , the equation  $x = y - B(x, x)$  has a unique solution  $\bar{x} \in E$  satisfying  $\|\bar{x}\| \leq \frac{1}{2\eta}$ .

### §3. Main results

Now, for  $T > 0$ , we say that  $u$  is a mild solution of NSE on  $[0, T]$  corresponding to a divergence-free initial datum  $u_0$  when  $u$  solves the integral equation

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u(\tau, \cdot) \otimes u(\tau, \cdot)) d\tau.$$

Above we have used the following notation: For a tensor  $F = (F_{ij})$  we define the vector  $\nabla \cdot F$  by  $(\nabla \cdot F)_i = \sum_{j=1}^d \partial_j F_{ij}$  and for two vectors  $u$  and  $v$ , we define their tensor product  $(u \otimes v)_{ij} = u_i v_j$ . The operator  $\mathbb{P}$  is the Helmholtz-Leray projection onto the divergence-free fields

$$(\mathbb{P}f)_j = f_j + \sum_{1 \leq k \leq d} R_j R_k f_k, \quad (3)$$

where  $R_j$  is the Riesz transforms defined as

$$R_j = \frac{\partial_j}{\sqrt{-\Delta}}, \quad \text{i. e.} \quad \widehat{R_j g}(\xi) = \frac{i\xi_j}{|\xi|} \hat{g}(\xi)$$

with  $\widehat{\cdot}$  denoting the Fourier transform. The heat kernel  $e^{t\Delta}$  is defined as

$$e^{t\Delta}u(x) = ((4\pi t)^{-d/2} e^{-|\cdot|^2/4t} * u)(x).$$

If  $X$  is a normed space and  $u = (u_1, u_2, \dots, u_d)$ ,  $u_i \in X$ ,  $1 \leq i \leq d$ , then we write

$$u \in X, \|u\|_X = \left( \sum_{i=1}^d \|u_i\|_X^2 \right)^{1/2}.$$

We define the auxiliary space  $\mathcal{K}_{q,r,T}^{s,\tilde{q}}$  which is made up by the functions  $u(t, x)$  such that

$$\|u\|_{\mathcal{K}_{q,r,T}^{s,\tilde{q}}} := \sup_{0 < t < T} t^{\frac{\alpha}{2}} \|u(t, \cdot)\|_{\dot{H}_{L^{\tilde{q},r}}^s} < \infty,$$

and

$$\lim_{t \rightarrow 0} t^{\frac{\alpha}{2}} \|u(t, \cdot)\|_{\dot{H}_{L^{\tilde{q},r}}^s} = 0, \quad (4)$$

where  $r, q, \tilde{q}, s$  being fixed constants satisfying

$$q, \tilde{q} \in (1, +\infty), r \geq 1, s \geq 0, \frac{s}{d} < \frac{1}{\tilde{q}} \leq \frac{1}{q} \leq \frac{s+1}{d},$$

and

$$\alpha = \alpha(q, \tilde{q}) = d \left( \frac{1}{q} - \frac{1}{\tilde{q}} \right).$$

In the case  $\tilde{q} = q$ , it is also convenient to define the space  $\mathcal{K}_{q,r,T}^{s,\tilde{q}}$  as the natural space  $L^\infty([0, T]; \dot{H}_{L^{q,r}}^s(\mathbb{R}^d))$  with the additional condition that its elements  $u(t, x)$  satisfy

$$\lim_{t \rightarrow 0} \|u(t, \cdot)\|_{\dot{H}_{L^{q,r}}^s} = 0. \quad (5)$$

**Remark 1.** The auxiliary space  $\mathcal{K}_{\tilde{q}} := \mathcal{K}_{d,\tilde{q},T}^{0,\tilde{q}}$  ( $\tilde{q} \geq d$ ) was introduced by Weissler and systematically used by Kato [20] and Cannone [5].

**Lemma 8.** *Let  $1 \leq r \leq \tilde{r} \leq \infty$ . Then we have the following imbedding maps*

$$\mathcal{K}_{q,1,T}^{s,\tilde{q}} \hookrightarrow \mathcal{K}_{q,r,T}^{s,\tilde{q}} \hookrightarrow \mathcal{K}_{q,\tilde{r},T}^{s,\tilde{q}} \hookrightarrow \mathcal{K}_{q,\infty,T}^{s,\tilde{q}}.$$

**Proof.** It is easily deduced from Lemma 1 (a) and the definition of  $\mathcal{K}_{q,r,T}^{s,\tilde{q}}$ .  $\square$



**Lemma 9.** If  $u_0 \in \dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$  with  $q > 1, r \geq 1, s \geq 0$ , and  $\frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}$  then for all  $\tilde{q}$  satisfying

$$\frac{s}{d} < \frac{1}{\tilde{q}} < \frac{1}{q},$$

we have

$$e^{t\Delta}u_0 \in \mathcal{K}_{q,1,\infty}^{s,\tilde{q}},$$

and the following imbedding map

$$\dot{H}_{L^{q,r}}^s(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{\tilde{q}}),\infty}(\mathbb{R}^d). \quad (6)$$

**Proof.** Before proving this lemma, we need to prove the following lemma.

**Lemma 10.** Suppose that  $u_0 \in L^{q,r}(\mathbb{R}^d)$  with  $1 \leq q \leq \infty$  and  $1 \leq r < \infty$ . Then  $\lim_{n \rightarrow \infty} \|\mathcal{X}_n u_0\|_{L^{q,r}} = 0$ , where  $n \in \mathbb{N}, \mathcal{X}_n(x) = 0$  for  $x \in \{x : |x| < n\} \cap \{x : |u_0(x)| < n\}$  and  $\mathcal{X}_n(x) = 1$  otherwise.

**Proof.** With  $\delta > 0$  being fixed, we have

$$\{x : |\mathcal{X}_n u_0(x)| > \delta\} \supseteq \{x : |\mathcal{X}_{n+1} u_0(x)| > \delta\}, \quad (7)$$

and

$$\bigcap_{n=0}^{\infty} \{x : |\mathcal{X}_n u_0(x)| > \delta\} = \{x : |u_0(x)| = +\infty\}. \quad (8)$$

We prove that

$$\mathcal{M}^d(\{x : |u_0(x)| = +\infty\}) = 0, \quad (9)$$

with  $\mathcal{M}^d$  being the Lebesgue measure in  $\mathbb{R}^d$ , assuming on the contrary

$$\mathcal{M}^d(\{x : |u_0(x)| = +\infty\}) > 0.$$

We have  $u_0^*(t) := \inf \{\tau : \mathcal{M}^d(\{x : |u_0(x)| > \tau\}) \leq t\} = +\infty$  for all  $t$  such that  $0 < t < \mathcal{M}^d(\{x : |u_0(x)| = +\infty\})$  and then  $\|u_0\|_{L^{q,r}} = +\infty$ , a contradiction.

Note that

$$\mathcal{M}^d(\{x : |\mathcal{X}_0 u_0(x)| > \delta\}) = \mathcal{M}^d(\{x : |u_0(x)| > \delta\}).$$

We prove that

$$\mathcal{M}^d(\{x : |u_0(x)| > \delta\}) < \infty, \quad (10)$$

assuming on the contrary

$$\mathcal{M}^d(\{x : |u_0(x)| > \delta\}) = \infty.$$

We have  $u_0^*(t) \geq \delta$  for all  $t > 0$ , from the definition of the Lorentz space, we get

$$\|u_0\|_{L^{q,r}} = \left( \int_0^\infty (t^{\frac{1}{q}} u_0^*(t))^r \frac{dt}{t} \right)^{\frac{1}{r}} \geq \left( \int_0^\infty (t^{\frac{1}{q}} \delta)^r \frac{dt}{t} \right)^{\frac{1}{r}} = \delta \left( \int_0^\infty t^{\frac{r}{q}-1} dt \right)^{\frac{1}{r}} = \infty,$$

a contradiction.

From (7), (8), (9), and (10), we infer that

$$\lim_{n \rightarrow \infty} \mathcal{M}^d(\{x : |\mathcal{X}_n u_0(x)| > \delta\}) = \mathcal{M}^d(\{x : |u_0(x)| = +\infty\}) = 0. \quad (11)$$

Set

$$u_n^*(t) = \inf \{ \tau : \mathcal{M}^d(\{x : |\mathcal{X}_n u_0(x)| > \tau\}) \leq t \}.$$

We have

$$u_n^*(t) \geq u_{n+1}^*(t). \quad (12)$$

Fixed  $t > 0$ . For any  $\epsilon > 0$ , from (11) it follows that there exists a number  $n_0 = n_0(t, \epsilon)$  large enough such that

$$\mathcal{M}^d(\{x : |\mathcal{X}_n u_0(x)| > \epsilon\}) \leq t, \forall n \geq n_0.$$

From this we deduce that

$$u_n^*(t) \leq \epsilon, \forall n \geq n_0,$$

therefore

$$\lim_{n \rightarrow \infty} u_n^*(t) = 0. \quad (13)$$

From (12) and (13), we apply Lebesgue's monotone convergence theorem to get

$$\lim_{n \rightarrow \infty} \|\mathcal{X}_n u_0\|_{L^{q,r}} = \lim_{n \rightarrow \infty} \left( \int_0^\infty (t^{\frac{1}{q}} u_n^*(t))^r \frac{dt}{t} \right)^{\frac{1}{r}} = 0. \quad \square$$

Now we return to prove Lemma 9. We prove that

$$\sup_{0 < t < \infty} t^{\frac{\alpha}{2}} \|e^{t\Delta} u_0\|_{\dot{H}_{L^{\tilde{q},1}}^s} \lesssim \|u_0\|_{\dot{H}_{L^{q,r}}^s}. \quad (14)$$

Set

$$\frac{1}{h} = 1 + \frac{1}{\tilde{q}} - \frac{1}{q}.$$

Applying Proposition 2.4 (c) in ([26], pp. 20) for convolution in the Lorentz spaces, we have

$$\begin{aligned} \|e^{t\Delta} u_0\|_{\dot{H}_{L^{\tilde{q},1}}^s} &= \|e^{t\Delta} \dot{\Delta}^s u_0\|_{L^{\tilde{q},1}} = \frac{1}{(4\pi t)^{d/2}} \|e^{-\frac{|\cdot|^2}{4t}} * \dot{\Delta}^s u_0\|_{L^{\tilde{q},1}} \lesssim \\ \frac{1}{t^{d/2}} \|e^{-\frac{|\cdot|^2}{4t}}\|_{L^{h,1}} \|\dot{\Delta}^s u_0\|_{L^{q,\infty}} &= t^{-\frac{\alpha}{2}} \|e^{-\frac{|\cdot|^2}{4t}}\|_{L^{h,1}} \|u_0\|_{\dot{H}_{L^{q,\infty}}^s} \lesssim t^{-\frac{\alpha}{2}} \|u_0\|_{\dot{H}_{L^{q,r}}^s}. \end{aligned}$$

We claim now that

$$\lim_{t \rightarrow 0} t^{\frac{\alpha}{2}} \|e^{t\Delta} u_0\|_{\dot{H}_{L^{\tilde{q},1}}^s} = 0.$$

From Lemma 10, we have

$$\lim_{n \rightarrow \infty} \left\| \mathcal{X}_{n,s} \dot{\Delta}^s u_0 \right\|_{L^{q,r}} = 0, \quad (15)$$

where  $\mathcal{X}_{n,s}(x) = 0$  for  $x \in \{x : |x| < n\} \cap \{x : |\dot{\Delta}^s u_0(x)| < n\}$  and  $\mathcal{X}_{n,s}(x) = 1$  otherwise. We have

$$\begin{aligned} t^{\frac{\alpha}{2}} \|e^{t\Delta} u_0\|_{\dot{H}_{L^{\tilde{q},1}}^s} &\leq \frac{t^{\frac{\alpha}{2} - \frac{d}{2}}}{(4\pi)^{d/2}} \left\| e^{-\frac{|\cdot|^2}{4t}} * (\mathcal{X}_{n,s} \dot{\Delta}^s u_0) \right\|_{L^{\tilde{q},1}} + \\ &\quad \frac{t^{\frac{\alpha}{2} - \frac{d}{2}}}{(4\pi)^{d/2}} \left\| e^{-\frac{|\cdot|^2}{4t}} * ((1 - \mathcal{X}_{n,s}) \dot{\Delta}^s u_0) \right\|_{L^{\tilde{q},1}}. \end{aligned} \quad (16)$$

For any  $\epsilon > 0$ , applying Proposition 2.4 (c) in ([26], pp. 20) and note that (15), we have

$$\begin{aligned} &\frac{t^{\frac{\alpha}{2} - \frac{d}{2}}}{(4\pi)^{d/2}} \left\| e^{-\frac{|\cdot|^2}{4t}} * (\mathcal{X}_{n,s} \dot{\Delta}^s u_0) \right\|_{L^{\tilde{q},1}} \\ &\leq C_1 \|e^{-\frac{|\cdot|^2}{4t}}\|_{L^{h,1}} \left\| \mathcal{X}_{n,s} \dot{\Delta}^s u_0 \right\|_{L^{q,\infty}} \leq C_2 \left\| \mathcal{X}_{n,s} \dot{\Delta}^s u_0 \right\|_{L^{q,r}} < \frac{\epsilon}{2}, \end{aligned} \quad (17)$$

for large enough  $n$ . Fixed one of such  $n$ , applying Proposition 2.4 (a) in ([26], pp. 20), we conclude that

$$\begin{aligned} &\frac{t^{\frac{\alpha}{2} - \frac{d}{2}}}{(4\pi)^{d/2}} \left\| e^{-\frac{|\cdot|^2}{4t}} * ((1 - \mathcal{X}_{n,s}) \dot{\Delta}^s u_0) \right\|_{L^{\tilde{q},1}} \\ &\leq C_3 t^{\frac{\alpha}{2} - \frac{d}{2}} \|e^{-\frac{|\cdot|^2}{4t}}\|_{L^1} \left\| (1 - \mathcal{X}_{n,s}) \dot{\Delta}^s u_0 \right\|_{L^{\tilde{q},1}} \\ &\leq C_4 t^{\frac{\alpha}{2}} \|e^{-\frac{|\cdot|^2}{4t}}\|_{L^1} \|n(1 - \mathcal{X}_{n,s})\|_{L^{\tilde{q},1}} = \\ &C_5 n t^{\frac{\alpha}{2}} \|(1 - \mathcal{X}_{n,s})\|_{L^{\tilde{q},1}} = C_6(n) t^{\frac{\alpha}{2}} < \frac{\epsilon}{2}, \end{aligned} \quad (18)$$

for small enough  $t > 0$ . From the estimates (16), (17), and (18) it follows that

$$t^{\frac{\alpha}{2}} \|e^{t\Delta} u_0\|_{\dot{H}_{L^{\tilde{q},1}}^s} \leq C_2 \left\| \mathcal{X}_{n,s} \dot{\Delta}^s u_0 \right\|_{L^{q,r}} + C_6(n) t^{\frac{\alpha}{2}} < \epsilon.$$

Finally, the embedding (6) is derived from the inequality (14), Lemma 1, and Lemma 6.

**Remark 2.** In the case  $s = 0$  and  $q = r = d$ , Lemma 11 is a generalization of Lemma 9 in ([8], p. 196).

In the following lemmas a particular attention will be devoted to study of the bilinear operator  $B(u, v)(t)$  defined by

$$B(u, v)(t) = \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u(\tau) \otimes v(\tau)) d\tau.$$

**Lemma 11.** *Let  $s, q \in \mathbb{R}$  be such that*

$$s \geq 0, q > 1, \text{ and } \frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}. \quad (19)$$

*Then for all  $\tilde{q}$  satisfying*

$$\frac{s}{d} < \frac{1}{\tilde{q}} < \min\left\{\frac{1}{2} + \frac{s}{2d}, \frac{1}{q}\right\}, \quad (20)$$

*the bilinear operator  $B(u, v)(t)$  is continuous from  $\mathcal{K}_{q, \tilde{q}, T}^{s, \tilde{q}} \times \mathcal{K}_{q, \tilde{q}, T}^{s, \tilde{q}}$  into  $\mathcal{K}_{q, 1, T}^{s, \tilde{q}}$  and the following inequality holds*

$$\|B(u, v)\|_{\mathcal{K}_{q, 1, T}^{s, \tilde{q}}} \leq C.T^{\frac{1}{2}(1+s-\frac{d}{q})} \|u\|_{\mathcal{K}_{q, \tilde{q}, T}^{s, \tilde{q}}} \|v\|_{\mathcal{K}_{q, \tilde{q}, T}^{s, \tilde{q}}}, \quad (21)$$

*where  $C$  is a positive constant independent of  $T$ .*

**Proof.** We have

$$\begin{aligned} \|B(u, v)(t)\|_{\dot{H}_{L^{\tilde{q}, 1}}^s} &\leq \int_0^t \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{\dot{H}_{L^{\tilde{q}, 1}}^s} d\tau = \\ &\int_0^t \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{\tilde{q}, 1}} d\tau. \end{aligned} \quad (22)$$

From the properties of the Fourier transform

$$\begin{aligned} &\left( e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right)_j^\wedge(\xi) = \\ &e^{-(t-\tau)|\xi|^2} \sum_{l, k=1}^d \left( \delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) (i \xi_l) \left( \dot{\Lambda}^s (u_l(\tau, \cdot) v_k(\tau, \cdot)) \right)_j^\wedge(\xi), \end{aligned}$$

and then

$$\begin{aligned} &\left( e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right)_j = \\ &\frac{1}{(t-\tau)^{\frac{d+1}{2}}} \sum_{l, k=1}^d K_{l, k, j} \left( \frac{\cdot}{\sqrt{t-\tau}} \right) * \left( \dot{\Lambda}^s (u_l(\tau, \cdot) v_k(\tau, \cdot)) \right), \end{aligned} \quad (23)$$

where

$$\widehat{K_{l,k,j}}(\xi) = \frac{1}{(2\pi)^{d/2}} \cdot e^{-|\xi|^2} \left( \delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) (i\xi_l).$$

Applying Proposition 11.1 ([26], p. 107) with  $|\alpha| = 1$  we see that the tensor  $K(x) = \{K_{l,k,j}(x)\}$  satisfies

$$|K(x)| \lesssim \frac{1}{(1 + |x|)^{d+1}}. \quad (24)$$

So, we can rewrite the equality (23) in the tensor form

$$\begin{aligned} e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) = \\ \frac{1}{(t-\tau)^{\frac{d+1}{2}}} K \left( \frac{\cdot}{\sqrt{t-\tau}} \right) * \left( \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right). \end{aligned} \quad (25)$$

Set

$$\frac{1}{r} = \frac{2}{\tilde{q}} - \frac{s}{d}, \quad \frac{1}{h} = \frac{s}{d} - \frac{1}{\tilde{q}} + 1. \quad (26)$$

From the inequalities (19) and (20), we can check that the following conditions are satisfied

$$1 < h, r < \infty \text{ and } \frac{1}{\tilde{q}} + 1 = \frac{1}{h} + \frac{1}{r}.$$

Applying Proposition 2.4 (c) in ([26], pp. 20) for convolution in the Lorentz spaces, we have

$$\begin{aligned} \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{\tilde{q},1}} \lesssim \\ \frac{1}{(t-\tau)^{\frac{d+1}{2}}} \left\| K \left( \frac{\cdot}{\sqrt{t-\tau}} \right) \right\|_{L^{h,1}} \left\| \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{r,\infty}}. \end{aligned} \quad (27)$$

Applying Lemma 4 we obtain

$$\begin{aligned} \left\| \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{r,\infty}} &\leq \left\| \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^r} = \|u(\tau, \cdot) \otimes v(\tau, \cdot)\|_{\dot{H}_r^s} \\ &\lesssim \|u(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \|v(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s}. \end{aligned} \quad (28)$$

Fom the inequalities (24) and (26) we infer that

$$\left\| K \left( \frac{\cdot}{\sqrt{t-\tau}} \right) \right\|_{L^{h,1}} = (t-\tau)^{\frac{d}{2h}} \|K\|_{L^{h,1}} \simeq (t-\tau)^{\frac{s}{2} - \frac{d}{2\tilde{q}} + \frac{d}{2}}. \quad (29)$$

From the inequalities (27), (28), and (29) we deduce that

$$\begin{aligned} & \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{\tilde{q},1}} \lesssim \\ & (t-\tau)^{\frac{s}{2} - \frac{d}{2\tilde{q}} - \frac{1}{2}} \|u(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \|v(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s}. \end{aligned} \quad (30)$$

From the estimates (22) and (30), and note that from the inequalities (19) and (20), we can check that  $\frac{s}{2} - \frac{d}{2\tilde{q}} - \frac{1}{2} > -1$  and  $\alpha = d(\frac{1}{q} - \frac{1}{\tilde{q}}) < 1$ , this gives the desired result

$$\begin{aligned} & \|B(u, v)(t)\|_{\dot{H}_{L^{\tilde{q},1}}^s} \lesssim \int_0^t (t-\tau)^{\frac{s}{2} - \frac{d}{2\tilde{q}} - \frac{1}{2}} \|u(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \|v(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} d\tau \lesssim \\ & \int_0^t (t-\tau)^{\frac{s}{2} - \frac{d}{2\tilde{q}} - \frac{1}{2}} \tau^{-\alpha} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|u(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|v(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} d\tau = \\ & \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|u(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|v(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \int_0^t (t-\tau)^{\frac{s}{2} - \frac{d}{2\tilde{q}} - \frac{1}{2}} \tau^{-\alpha} d\tau \simeq \\ & t^{-\frac{\alpha}{2}} t^{\frac{1}{2}(1+s-\frac{d}{q})} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|u(\eta, \cdot)\|_{\dot{H}_{L^{\tilde{q},\tilde{q}}}^s} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|v(\eta, \cdot)\|_{\dot{H}_{L^{\tilde{q},\tilde{q}}}^s}. \end{aligned} \quad (31)$$

Let us now check the validity of the condition (4) for the bilinear term  $B(u, v)(t)$ . Indeed, we have

$$\lim_{t \rightarrow 0} t^{\frac{\alpha}{2}} \|B(u, v)(t)\|_{\dot{H}_{L^{\tilde{q},1}}^s} = 0,$$

whenever

$$\lim_{t \rightarrow 0} t^{\frac{\alpha}{2}} \|u(t, \cdot)\|_{\dot{H}_{\tilde{q}}^s} = \lim_{t \rightarrow 0} t^{\frac{\alpha}{2}} \|v(t, \cdot)\|_{\dot{H}_{\tilde{q}}^s} = 0.$$

The estimate (21) is now deduced from the inequality (31).  $\square$

**Remark 3.** In the case  $s = 0$  and  $q = d$ , Lemma 9 is a generalization of Lemma 10 in ([8], p. 196).

**Lemma 12.** *Let  $s, q \in \mathbb{R}$  be such that*

$$s \geq 0, q > 1, \text{ and } \frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}. \quad (32)$$

*Then for all  $\tilde{q}$  satisfying*

$$\frac{1}{2} \left( \frac{1}{q} + \frac{s}{d} \right) < \frac{1}{\tilde{q}} < \min \left\{ \frac{1}{2} + \frac{s}{2d}, \frac{1}{q} \right\}, \quad (33)$$

*the bilinear operator  $B(u, v)(t)$  is continuous from  $\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}} \times \mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}$  into  $\mathcal{K}_{q,1,T}^{s,q}$  and the following inequality holds*

$$\|B(u, v)\|_{\mathcal{K}_{q,1,T}^{s,q}} \leq C.T^{\frac{1}{2}(1+s-\frac{d}{q})} \|u\|_{\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}} \|v\|_{\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}}, \quad (34)$$

*where  $C$  is a positive constant independent of  $T$ .*

**Proof.** Set

$$\frac{1}{r} = \frac{2}{\tilde{q}} - \frac{s}{d}, \quad \frac{1}{h} = 1 + \frac{1}{q} - \frac{2}{\tilde{q}} + \frac{s}{d}. \quad (35)$$

From the inequalities (32) and (33), we can check that  $h$  and  $r$  satisfy

$$1 < h, r < \infty \text{ and } \frac{1}{q} + 1 = \frac{1}{h} + \frac{1}{r}.$$

From the equality (25), applying Proposition 2.4 (c) in ([26], pp. 20), we obtain

$$\begin{aligned} & \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \dot{\Lambda}^s(u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{q,1}} \lesssim \\ & \frac{1}{(t-\tau)^{\frac{d+1}{2}}} \left\| K\left(\frac{\cdot}{\sqrt{t-\tau}}\right) \right\|_{L^{h,1}} \left\| \dot{\Lambda}^s(u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{r,\infty}}. \end{aligned} \quad (36)$$

Applying Lemma 4, we have

$$\begin{aligned} & \left\| \dot{\Lambda}^s(u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{r,\infty}} \leq \left\| \dot{\Lambda}^s(u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^r} \\ & \lesssim \|u(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \|v(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s}. \end{aligned} \quad (37)$$

From the inequalities (24) and (35) it follows that

$$\left\| K\left(\frac{\cdot}{\sqrt{t-\tau}}\right) \right\|_{L^{h,1}} = (t-\tau)^{\frac{d}{2h}} \|K\|_{L^{h,1}} \simeq (t-\tau)^{\frac{d}{2} + \frac{d}{2q} - \frac{d}{\tilde{q}} + \frac{s}{2}}. \quad (38)$$

From the estimates (36), (37), (38) we deduce that

$$\begin{aligned} & \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{\dot{H}_{L^{q,1}}^s} \lesssim (t-\tau)^{\frac{d}{2q} - \frac{d}{\tilde{q}} + \frac{s}{2} - \frac{1}{2}} \|u(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \|v(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \\ & = (t-\tau)^{\alpha + \frac{s}{2} - \frac{d}{2q} - \frac{1}{2}} \|u(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \|v(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s}. \end{aligned}$$

From the inequalities (32) and (33), we can check that  $\alpha + \frac{s}{2} - \frac{d}{2q} - \frac{1}{2} > -1$  and  $\alpha = d(\frac{1}{q} - \frac{1}{\tilde{q}}) < 1$ , this gives the desired result

$$\begin{aligned} & \|B(u, v)(t)\|_{\dot{H}_{L^{q,1}}^s} \lesssim \int_0^t (t-\tau)^{\alpha + \frac{s}{2} - \frac{d}{2q} - \frac{1}{2}} \|u(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \|v(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} d\tau \lesssim \\ & \int_0^t (t-\tau)^{\alpha + \frac{s}{2} - \frac{d}{2q} - \frac{1}{2}} \tau^{-\alpha} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|u(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|v(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} d\tau = \\ & \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|u(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|v(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \int_0^t (t-\tau)^{\alpha + \frac{s}{2} - \frac{d}{2q} - \frac{1}{2}} \tau^{-\alpha} d\tau \simeq \\ & t^{\frac{1}{2}(1+s-\frac{d}{q})} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|u(\eta, \cdot)\|_{\dot{H}_{L^{\tilde{q}, \tilde{q}}}^s} \cdot \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|v(\eta, \cdot)\|_{\dot{H}_{L^{\tilde{q}, \tilde{q}}}^s}. \end{aligned} \quad (39)$$

Let us now check the validity of the condition (5) for the bilinear term  $B(u, v)(t)$ . Indeed, we have

$$\lim_{t \rightarrow 0} \|B(u, v)(t)\|_{\dot{H}_{L^{q,1}}^s} = 0$$

whenever

$$\lim_{t \rightarrow 0} t^{\frac{s}{2}} \|u(t, \cdot)\|_{\dot{H}_{\tilde{q}}^s} = \lim_{t \rightarrow 0} t^{\frac{s}{2}} \|v(t, \cdot)\|_{\dot{H}_{\tilde{q}}^s} = 0.$$

The estimate (34) is now deduced from the inequality (39).  $\square$

Combining Theorem 1 with Lemmas 7, 9, 11, 12, we obtain the following existence result.

**Theorem 2.** *Let  $s, q$ , and  $r \in \mathbb{R}$  be such that*

$$s \geq 0, q > 1, r \geq 1, \text{ and } \frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}. \quad (40)$$

(a) *For all  $\tilde{q}$  satisfying*

$$\frac{1}{2} \left( \frac{1}{q} + \frac{s}{d} \right) < \frac{1}{\tilde{q}} < \min \left\{ \frac{1}{2} + \frac{s}{2d}, \frac{1}{q} \right\}, \quad (41)$$

*there exists a positive constant  $\delta_{s,q,\tilde{q},d}$  such that for all  $T > 0$  and for all  $u_0 \in \dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$  with  $\operatorname{div}(u_0) = 0$  satisfying*

$$T^{\frac{1}{2}(1+s-\frac{d}{q})} \sup_{0 < t < T} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})} \|e^{t\Delta} u_0\|_{\dot{H}_{\tilde{q}}^s} \leq \delta_{s,q,\tilde{q},d}, \quad (42)$$

*NSE has a unique mild solution  $u \in \mathcal{K}_{q,1,T}^{s,\tilde{q}} \cap L^\infty([0, T]; \dot{H}_{L^{q,r}}^s)$ . In particular, for arbitrary  $u_0 \in \dot{H}_{L^{q,r}}^s$  with  $\operatorname{div}(u_0) = 0$ , there exists  $T(u_0)$  small enough such that the inequality (42) holds.*

(b) *If  $1 < q \leq d$ , and  $s = \frac{d}{q} - 1$  then for any  $\tilde{q}$  be such that*

$$\frac{1}{q} - \frac{1}{2d} < \frac{1}{\tilde{q}} < \min \left\{ \frac{1}{2} + \frac{1}{2q} - \frac{1}{2d}, \frac{1}{q} \right\},$$

*there exists a positive constant  $\sigma_{q,\tilde{q},d}$  such that if  $\|u_0\|_{\dot{B}_{\tilde{q}}^{\frac{d}{q}-1,\infty}} \leq \sigma_{q,\tilde{q},d}$*

*and  $T = \infty$  then the inequality (42) holds.*

**Proof.** From Lemmas 11 and 8, the bilinear operator  $B(u, v)(t)$  is continuous from  $\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}} \times \mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}$  into  $\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}$  and we have the inequality

$$\|B(u, v)\|_{\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}} \leq \|B(u, v)\|_{\mathcal{K}_{q,1,T}^{s,\tilde{q}}} \leq C_{s,q,\tilde{q},d} T^{\frac{1}{2}(1+s-\frac{d}{q})} \|u\|_{\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}} \|v\|_{\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}},$$



where  $C_{s,q,\tilde{q},d}$  is a positive constant independent of  $T$ . From Theorem 1 and the above inequality, we deduce following: for any  $u_0 \in \dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$  such that

$$\operatorname{div}(u_0) = 0, \quad T^{\frac{1}{2}(1+s-\frac{d}{q})} \sup_{0 < t < T} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})} \|e^{t\Delta} u_0\|_{\dot{H}_{\tilde{q}}^s} \leq \frac{1}{4C_{s,q,\tilde{q},d}},$$

NSE has a mild solution  $u$  on the interval  $(0, T)$  so that

$$u \in \mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}. \quad (43)$$

Lemma 12 and the relation (43) imply that

$$B(u, u) \in \mathcal{K}_{q,1,T}^{s,q} \subseteq \mathcal{K}_{q,r,T}^{s,q} \subseteq L^\infty([0, T]; \dot{H}_{L^{q,r}}^s).$$

On the other hand, from Lemma 7, we have  $e^{t\Delta} u_0 \in L^\infty([0, T]; \dot{H}_{L^{q,r}}^s)$ .

Therefore

$$u = e^{t\Delta} u_0 - B(u, u) \in L^\infty([0, T]; \dot{H}_{L^{q,r}}^s).$$

From Lemma 9 and Lemma 11, we deduce that  $u \in \mathcal{K}_{q,1,T}^{s,\tilde{q}}$ .

From the definition of  $\mathcal{K}_{q,r,T}^{s,\tilde{q}}$  and Lemma 9, we deduce that the left-hand side of the inequality (42) converges to 0 when  $T$  tends to 0. Therefore the inequality (42) holds for arbitrary  $u_0 \in \dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$  when  $T(u_0)$  is small enough.

(b) From Lemma 6, the two quantities

$$\|u_0\|_{\dot{B}_{\tilde{q}}^{\frac{d}{q}-1,\infty}} \quad \text{and} \quad \sup_{0 < t < \infty} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})} \|e^{t\Delta} u_0\|_{\dot{H}_{\tilde{q}}^{\frac{d}{q}-1}}$$

are equivalent, then there exists a positive constant  $\sigma_{q,\tilde{q},d}$  such that if  $\|u_0\|_{\dot{B}_{\tilde{q}}^{\frac{d}{q}-1,\infty}} \leq \sigma_{q,\tilde{q},d}$  and  $T = \infty$  then the inequality (42) holds.  $\square$

**Remark 4.** In the case when the initial data belong to the critical Sobolev-Lorentz spaces  $\dot{H}_{L^{q,r}}^{\frac{d}{q}-1}(\mathbb{R}^d)$ ,  $(1 < q \leq d, r \geq 1)$ , from Theorem 2 (b), we get the existence of global mild solutions in the spaces  $L^\infty([0, \infty); \dot{H}_{L^{q,r}}^{\frac{d}{q}-1}(\mathbb{R}^d))$  when the norm of the initial value in the Besov spaces  $\dot{B}_{\tilde{q}}^{\frac{d}{q}-1,\infty}(\mathbb{R}^d)$  is small enough. Note that a function in  $\dot{H}_{L^{q,r}}^{\frac{d}{q}-1}(\mathbb{R}^d)$  can be arbitrarily large in the  $\dot{H}_{L^{q,r}}^{\frac{d}{q}-1}(\mathbb{R}^d)$  norm but small in the  $\dot{B}_{\tilde{q}}^{\frac{d}{q}-1,\infty}(\mathbb{R}^d)$  norm. This is deduced from the following imbedding maps (see Lemma 9)

$$\dot{H}_{L^{q,r}}^{\frac{d}{q}-1}(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{\frac{d}{q}-1,\infty}(\mathbb{R}^d), \quad \left(\frac{1}{q} - \frac{1}{d} < \frac{1}{\tilde{q}} < \frac{1}{q}\right).$$

This result is stronger than that of Cannone. In particular, when  $q = r = d, s = 0$ , we get back the Cannone theorem (Theorem 1.1 in [5]).

Next, we consider the super-critical indexes  $s > \frac{d}{q} - 1$ .

**Theorem 3.** *Let*

$$s \geq 0, q > 1, r \geq 1, \text{ and } \frac{s}{d} < \frac{1}{q} < \frac{s+1}{d}.$$

*Then for any  $\tilde{q}$  be such that*

$$\frac{1}{2} \left( \frac{1}{q} + \frac{s}{d} \right) < \frac{1}{\tilde{q}} < \min \left\{ \frac{1}{2} + \frac{s}{2d}, \frac{1}{q} \right\},$$

*there exists a positive constant  $\delta_{s,q,\tilde{q},d}$  such that for all  $T > 0$  and for all  $u_0 \in \dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$  with  $\operatorname{div}(u_0) = 0$  satisfying*

$$T^{\frac{1}{2}(1+s-\frac{d}{q})} \|u_0\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{q}),\infty}} \leq \delta_{s,q,\tilde{q},d},$$

*NSE has a unique mild solution  $u \in \mathcal{K}_{q,1,T}^{s,\tilde{q}} \cap L^\infty([0, T]; \dot{H}_{L^{q,r}}^s)$ .*

**Proof.** Applying Lemma 6, the two quantities  $\|u_0\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{q}),\infty}}$  and  $\sup_{0 < t < \infty} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})} \|e^{t\Delta} u_0\|_{\dot{H}_{\tilde{q}}^s}$  are equivalent. Thus

$$\sup_{0 < t < T} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})} \|e^{t\Delta} u_0\|_{\dot{H}_{\tilde{q}}^s} \lesssim \|u_0\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{q}),\infty}},$$

the theorem is proved by applying the above inequality and Theorem 2.  $\square$

**Remark 5.** In the case when the initial data belong to the Sobolev-Lorentz spaces  $\dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$ , ( $q > 1, r \geq 1, s \geq 0$ , and  $\frac{d}{q} - 1 < s < \frac{d}{q}$ ), we obtain the existence of mild solutions in the spaces  $L^\infty([0, T]; \dot{H}_{L^{q,r}}^s(\mathbb{R}^d))$  for any  $T > 0$  when the norm of the initial value in the Besov spaces  $\dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{q}),\infty}(\mathbb{R}^d)$  is small enough. Note that a function in  $\dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$  can be arbitrarily large in the  $\dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$  norm but small in  $\dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{q}),\infty}(\mathbb{R}^d)$  norm. This is deduced from the following imbedding maps (see Lemma 9)

$$\dot{H}_{L^{q,r}}^s(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{q}),\infty}(\mathbb{R}^d), \quad \left( \frac{s}{d} < \frac{1}{\tilde{q}} < \frac{1}{q} \right).$$

Applying Theorem 3 for  $q > d, r = q$  and  $s = 0$ , we get the following proposition which is stronger than the result of Cannone and Meyer ([4], [7]). In particular, we obtained a result that is stronger than that of Cannone and Meyer but under a much weaker condition on the initial data.

**Proposition 1.** *Let  $q > d$ . Then for any  $\tilde{q}$  be such that*

$$q < \tilde{q} < 2q,$$

*there exists a positive constant  $\delta_{q,\tilde{q},d}$  such that for all  $T > 0$  and for all  $u_0 \in L^q(\mathbb{R}^d)$  with  $\operatorname{div}(u_0) = 0$  satisfying*

$$T^{\frac{1}{2}(1-\frac{d}{q})} \|u_0\|_{\dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-\frac{d}{q},\infty}} \leq \delta_{q,\tilde{q},d}, \quad (44)$$

*NSE has a unique mild solution  $u \in \mathcal{K}_{q,1,T}^{0,\tilde{q}} \cap L^\infty([0, T]; L^q)$ .*

**Remark 6.** If in (44) we replace the  $\dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-\frac{d}{q},\infty}$  norm by the  $L^q$  norm then we get the assumption made in ([4], [7]). We show that the condition (44) is weaker than the condition in ([4], [7]). In Remark 5 we have showed that

$$L^q(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-\frac{d}{q},\infty}(\mathbb{R}^d), (\tilde{q} > q \geq d),$$

but these two spaces are different. Indeed, we have  $|x|^{-\frac{d}{q}} \notin L^q(\mathbb{R}^d)$ . On the other hand by using Lemma 6, we can easily prove that  $|x|^{-\frac{d}{q}} \in \dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-\frac{d}{q},\infty}(\mathbb{R}^d)$  for all  $\tilde{q} > q$ .

Applying Theorem 3 for  $q = r = 2, \frac{d}{2} - 1 < s < \frac{d}{2}$ , we get the following proposition which is stronger than the results of Chemin in [9] and Cannone in [4]. In particular, we obtained the result that is stronger than that of Chemin and Cannone but under a much weaker condition on the initial data.

**Proposition 2.** *Let  $\frac{d}{2} - 1 < s < \frac{d}{2}$ . Then for any  $\tilde{q}$  be such that*

$$\frac{1}{2} \left( \frac{1}{2} + \frac{s}{d} \right) < \frac{1}{\tilde{q}} < \frac{1}{2},$$

*there exists a positive constant  $\delta_{s,\tilde{q},d}$  such that for all  $T > 0$  and for all  $u_0 \in \dot{H}^s(\mathbb{R}^d)$  with  $\operatorname{div}(u_0) = 0$  satisfying*

$$T^{\frac{1}{2}(1+s-\frac{d}{2})} \|u_0\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{2}-\frac{d}{\tilde{q}}),\infty}} \leq \delta_{s,\tilde{q},d}, \quad (45)$$

*NSE has a unique mild solution  $u \in \mathcal{K}_{2,1,T}^{s,\tilde{q}} \cap L^\infty([0, T]; \dot{H}^s)$ .*

**Remark 7.** If in (45) we replace the  $\dot{B}_{\tilde{q}}^{s-(\frac{d}{2}-\frac{d}{\tilde{q}}),\infty}$  norm by the  $\dot{H}^s(\mathbb{R}^d)$  norm then we get the assumption made in ([9], [4]). We show that the condition (45) is weaker than the condition in ([9], [4]). In Remark 5 we showed that

$$\dot{H}^s(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{s-(\frac{d}{2}-\frac{d}{\tilde{q}}),\infty}, \quad \frac{1}{2} \left( \frac{1}{2} + \frac{s}{d} \right) < \frac{1}{\tilde{q}} < \frac{1}{2},$$

but that these two spaces are different. Indeed, we have  $\dot{\Lambda}^{-s}|\cdot|^{-\frac{d}{2}} \notin \dot{H}^s(\mathbb{R}^d)$ , on the other hand by using Lemma 6, we easily prove that  $\dot{\Lambda}^{-s}|\cdot|^{-\frac{d}{2}} \in \dot{B}_{\tilde{q}}^{s-(\frac{d}{2}-\frac{d}{\tilde{q}}),\infty}(\mathbb{R}^d)$  for all  $\tilde{q} > 2$ .

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